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ON THE EQUILIBRIUM AND STABILITY OF DISCRETE ONE-WAY STRUCTURAL SYSTEMS†

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Abstract—One-way structural systems are discussed in the context of the generalized-coordinate approach to discrete systems. In these terms a general procedure for the treatment of such systems is initiated, and conditions for equilibrium and stability of equilibrium are developed. The application of the general theory to particular problems is demonstrated in three simple structural models with two degrees of freedom.

1. INTRODUCTION

AN INCREASING amount of interest is currently being shown in the analysis of structural systems with "one-way" characteristics. These are structures whose deformations under load are restricted by rigid barriers from going into certain regions. The term "one-way" refers to the fact that when restrained by such a barrier the local deformations of a system can only move in the general direction away from the barrier.

The most prolific field of study to date has been the buckling of a thin circular elastic ring loaded circumferentially within a rigid cavity. The initial investigation of this problem was by Lo *et al.* [1] using a Tsien "equal-energy" criterion to determine "buckling loads". A later study by Elkon *et al.* [2] solved the problem more satisfactorily, and both Chan and McMinn [3] and Britvec [4] reworked the identical problem, Britvec working in the context of thermal loading. Further studies have been carried out for similar problems, under different loading conditions and with initial imperfections in the ring and in the rigid boundary. A logical extension of the work on rings was the analysis by Tong and Pian [5] of the buckling of a spherical shell loaded within a rigid spherical cavity. In all of this work the system under consideration has been treated as a continuum and the treatment has been in "vectorial" rather than "variational" terms. Two studies unconnected with rings have been carried out by Allan [6, 7], on one-way situations of axially loaded elastic columns. The first concerns both finite and infinite laterally loaded columns prevented from deflecting to one side of their initial axes by a rigid barrier, and the second concerns a similar problem of columns with elastic foundations. The analyses presented are once again particular vectorial treatments in continuum terms.

The problems so far described all have a high degree of symmetry, and if real engineering problems of one-way systems are to extend to more complex, less symmetrical types of structure it is inevitable that continuum analysis will have to give way as a practical method of solution to treatments which represent a continuum approximately as a discrete system. Examples of such methods are the Rayleigh–Ritz, Galerkin and finite-element procedures, which are well-known and powerful tools in the analysis of practical structural systems.

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These methods use the potential energy of a system expressed as a function of a finite number of generalised coordinates which completely describe the possible displacements of the system. For normal systems, without one-way barriers to their displacements, the basic conditions in potential-energy terms for statical equilibrium and stability of equilibrium are well established, having their roots in the classical mechanics of the eighteenth century. Recent work on the general theory of discrete systems is thus almost entirely confined to "non-linear" studies concerned chiefly with equilibrium-path configurations about positions of critical equilibrium. Notable among these is the work of Thompson, [8, 9], on the theory of snapping- and bifurcation-point configurations in the context of generalized coordinates.

In the field of the structural mechanics of one-way systems there has been no specific study in general terms until the present, either in the continuum or discrete context. No general rules are available for the treatment of such systems by energy methods, and even the basic conditions for equilibrium and stability have not so far been postulated. Without such basic information the engineer is unable to undertake discretized treatments of complex continuum problems. The object of the present work is to provide a formal basis for the treatment of one-way structural systems in discrete terms, and to develop the basic criteria for equilibrium and stability in terms of generalized coordinates. It will be necessary in the course of the work to define a specialized notation to help in dealing with arbitrarily large determinants.

2. DISCUSSION

We consider a discrete conservative structural system whose configurations are completely described by the values of n generalized coordinates Q_i and whose loading state is described by the value of a single load-parameter Λ. We define a "free system" as one whose generalized coordinates Q_i are all free to take any independent small increments, either positive or negative. We also define a "one-way system" as one whose generalized coordinates are prevented by impassable boundaries to their configuration-space from taking increments in certain directions.

Boundaries to a system's permissible configuration-space are caused by the effect of physical rigid boundaries to the motion of the physical system. Any physical boundary to the system's displacements will cause boundaries in configuration-space, separating regions in which the system's generalized coordinates are free to take values from prohibited regions. It does not necessarily follow that a continuous physical boundary will give rise to a continuous boundary in configuration-space; it may, in fact, cause several continuous boundaries in configuration-space, which may intersect each other in discontinuities. We can find all of these configuration-space boundaries for a system by considering the limiting configurations of the system caused by its physical boundaries, and can express each of these configuration-space boundaries as an equation of the form $F^j(Q_1, Q_2, \ldots) = 0$. We can also fix the function $F^j(Q_i)$ so that its value is always positive when the configuration lies on the permissible side of the jth boundary, and negative for any configuration within the space prohibited by the boundary. If there are M such configuration-space boundaries then each of the M boundary functions $F^j(Q_i)$ must be greater than or equal to zero for any permissible configuration. Each boundary function whose value is zero indicates that the system's configuration lies upon the corresponding boundary. The boundaries to a system's configuration-space are assumed fixed in that space, and since the physical boundaries are rigid they are in no way dependent upon the current value of the loading parameter.

We consider a conservative structural system whose total potential energy has the form,

$$V(Q_i, \Lambda) = U(Q_i) - \Lambda\varepsilon(Q_i), \tag{1}$$

which is the strain energy of distortion of the system less the work done on the system by its load in bringing about the distortion. The condition for statical equilibrium of such a system in the form due to Fourier [10] may be stated as, "a system is in equilibrium if, for any permissible virtual displacement from its position at constant load, the first variation of its potential energy is *greater than or equal to* zero". This statement implies that the generalized forces acting on the system when it is displaced infinitesimally in any permissible direction must either act towards the original position or be zero. This principle may be demonstrated conceptually by considering states of two-degrees-of-freedom systems around positions of interest. Figures 1(a)–(h), shows certain of these states, in which the potential-energy fields around the positions of interest are shown as "contours" of equal potential energy and the dashed arrows show the directions of decrease of potential energy between contours (or, the directions of the generalized forces). Continuous, well-behaved potential energy forms are assumed in these contour-pictures as they are throughout the analysis. With regard to Fig. 1 it is evident that the points X show free equilibrium configurations in (a) and (b), coincident free and one-way equilibrium configurations in (c) and (d), genuine one-way equilibrium configurations in (e), (f) and (h), and no equilibrium in (g).

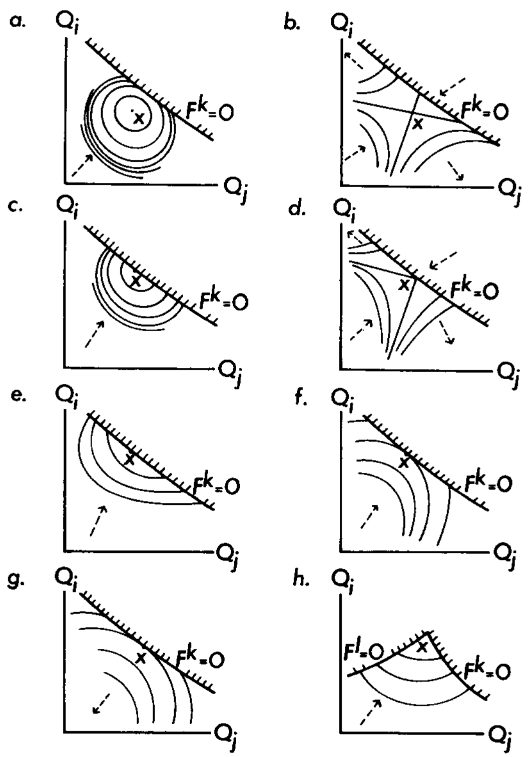

FIG. 1. Potential energy contours around points of interest in two-degrees-of-freedom one-way systems.

The definition of stability for a statical equilibrium position is that if any possible small displacement is given to the system and the displacing force is then released, the system will remain indefinitely in "the vicinity" of its equilibrium position. In more realistic terms, if the system has damping of a "viscous" type, it will eventually return to rest at its equilibrium position. This definition is quite adequate for one-way systems if we restrict all displacements to permissible zones. The Dirichlet criterion, which says that we must have a real local potential energy minimum for stability of equilibrium is still valid if we do not exclude discontinuous minima. Hence, Figs. 1(a), (c), (e) and (h) show stable equilibrium positions at X [where Fig. 1(h) shows a wholly discontinuous potential energy minimum] and Figs. 1(b), (d) and (f) show unstable equilibrium positions at X.

3. EQUILIBRIUM CONDITIONS

The variation of the potential energy of a system in a given state of loading can be expanded in power-series form about a given configuration $Q_i = Q_i^F$, in terms of the increments q_i of the n generalized coordinates Q_i.

$$V(Q_i, \Lambda) = V(Q_i^F, \Lambda) + \sum_{i=1}^{n} V_i(Q_k^F, \Lambda) q_i + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}(Q_k^F, \Lambda) q_i q_j + \ldots \tag{2}$$

Throughout this paper the subscript i appended to a function of the generalized coordinates denotes a partial differentiation by the appropriate generalized coordinate Q_i. A summed series is always preceded by Σ in order to distinguish series from single terms.

In its configuration $Q_i = Q_i^F$ the system lies upon m of its total of M boundaries. We call these m boundaries the system's "effective boundaries" in its given configuration. If we assume for the present that $m < n$, the system is free to change its configuration without moving away from any of these boundaries. To show that such a system is in equilibrium it will be necessary and sufficient to show that:

(a) For any incremental displacement which does not take the system away from any of its m effective boundaries the first variation of its potential energy is zero.

(b) For any incremental displacement which takes the system away from any or all of its effective boundaries the first variation of its potential energy is greater than or equal to zero.

We consider first an arbitrary incremental displacement of the former, type (a). The first variation of the potential energy is given by,

$$\delta V = \sum_{i=1}^{n} V_i q_i = 0. \tag{3}$$

In this case, the first variation of each of the effective boundary functions $F^r(Q_i)$ remains zero throughout the displacement. In order to apply these kinematic constraints we multiply each boundary function $F^r(Q_i)$ by a Lagrange multiplier λ^r and add its first variation to (3).

$$\delta W = \delta V + \delta \left[\sum_{r=1}^{m} \lambda^r F^r \right], = \sum_{i=1}^{n} \left[V_i + \sum_{r=1}^{m} \lambda^r F_i^r \right] q_i. \tag{4}$$

Having amended the first variation of the potential energy in this fashion we are then entitled to treat each of the q_i as independent and arbitrary. We can, thus, separate out the

n equations,

$$V_i + \sum_{r=1}^{m} \lambda^r F_i^r = 0, \tag{5}$$

in which the m Lagrange multipliers λ^r are the unknown quantities. Considering only the first m of the equations (5) we can express this system of equations in the notation defined in the Appendix as,

$$[\phi^T] . [\lambda^r] = [-V_s] \tag{6}$$

Here $[\phi^T]$ denotes the transpose of the matrix $[\phi]$. Noting that the determinant of a matrix is identical to the determinant of its transpose we can now solve (6) for each of the Lagrange multipliers.

$$\lambda^j = \frac{|\phi(-V_s)|^j}{|\phi|}, \qquad j = 1, \ldots, m. \tag{7}$$

These solutions for λ^j are now inserted into the remaining $(n-m)$ equations (5), giving

$$V_p + \sum_{j=1}^{m} F_p^j \frac{|\phi(-V_s)|^j}{|\phi|} = 0, \tag{8}$$

where p is any integer from $(m+1)$ to n. These equations can be expressed as,

$$V_p|\phi| - \sum_{i=1}^{m} V_i |\phi(F_p^j)|_i = 0, \tag{9}$$

which is equivalent to,

$$|\Omega_p| = 0, \qquad p = (m+1), \ldots, n. \tag{10}$$

Consider now an arbitrary incremental displacement which takes the system away from any or all of its m effective boundaries. For any such displacement we know by definition that the first variation of each boundary function $F^j(Q_i)$ must be greater than or equal to zero, and we say that the first variation of the potential energy must be greater than or equal to zero for equilibrium. Since there is no continuous potential energy minimum involved in this process we cannot use a Lagrange multiplier method to include the constraining conditions. Instead we use a simple substitution process. An incremental movement of the system away from some or all of its effective boundaries may be described by the values of the m quantities δF^j together with the values of any $(n-m)$ of the q_i. Separating the first m of the q_i from the last $(n-m)$, we can write m equations

$$\sum_{r=1}^{m} F_r^j q_r = \delta F^j - \sum_{p=(m+1)}^{n} F_p^j q_p, \qquad j = 1, \ldots, m. \tag{11}$$

The solutions for the first m of the q_i are,

$$q_r = \frac{\left| \phi \left(\delta F^j - \sum_{p=(m+1)}^{n} F_p^j q_p \right) \right|_r}{|\phi|}. \tag{12}$$

This may be rewritten,

$$q_r = \frac{|\phi(\delta F^j)|_r}{|\phi|} - \sum_{p=(m+1)}^{n} \frac{|\phi(F_p^j)|_r q_p}{|\phi|}. \tag{13}$$

All displacements are characterized by the property that the first variation of the potential energy must be greater than or equal to zero. The first variation is,

$$\begin{aligned}\delta V &= \sum_{i=1}^{n} V_i q_i, \\ &= \sum_{r=1}^{m} \sum_{p=(m+1)}^{n} \frac{[-V_r q_p |\phi(F_p^j)|_r]}{|\phi|} + \sum_{p=(m+1)}^{n} V_p q_p \\ &\quad + \sum_{r=1}^{m} \frac{V_r |\phi(\delta F^j)|_r}{|\phi|}.\end{aligned} \tag{14}$$

This is equivalent to,

$$\delta V = \sum_{p=(m+1)}^{n} q_p \frac{|\Omega_p|}{|\phi|} + \sum_{r=1}^{m} \frac{V_r |\phi(\delta F^j)|_r}{|\phi|}. \tag{15}$$

From (10) we know that $|\Omega_p|$ is zero, so that (15) can be rewritten,

$$\delta V = \sum_{j=1}^{m} \delta F^j \frac{|\phi(V_r)|^j}{|\phi|}. \tag{16}$$

We have defined that each permissible δF^j must be greater than or equal to zero. We also know that the value of each $|\phi(V_r)|^j/|\phi|$ is constant at fixed load and a given configuration, and is independent of the incremental displacements q_i. Hence the condition for δV to be greater than or equal to zero for arbitrary positive δF^j is that each of the coefficients $|\phi(V_r)|^j/|\phi|$ must be greater than or equal to zero.

The conditions for equilibrium of a system at a fixed loading state and in a given configuration on m effective boundaries are then,

$$|\Omega_p| = 0, \qquad p = (m+1), \ldots, n$$

and,

$$\frac{|\phi(V_r)|^j}{|\phi|} \geq 0, \qquad j = 1, \ldots, m. \tag{17}$$

In cases where $m = n$ it is easily seen that (15) reduces to (16) without reference to $|\Omega_p|$ so that the n equilibrium conditions are then all of the form,

$$\frac{|\phi(V_r)|^j}{|\phi|} \geq 0, \qquad j = 1, \ldots, n. \tag{18}$$

In cases where $m > n$ it is necessary to show that the condition (18) is satisfied for at least one combination of n of the m boundaries for equilibrium.

It is helpful conceptually if we consider each of the expressions $|\phi(V_r)|^j/|\phi|$ as being equivalent to the generalized force of reaction between the system and its jth boundary at its given configuration. If one of these forces is zero it is clear that the boundary represented by $F^j = 0$ has no effect on the equilibrium state other than to limit the directions in which the system can be disturbed.

Then if, say, k of these forces are zero we effectively have a system in equilibrium on $(m-k)$ boundaries, to which k boundaries have been added without disturbing the state. These k additional boundaries, however, still restrict the directions in which the system can be disturbed from its equilibrium position, and thus cannot be completely ignored in considering the stability of an equilibrium state. This type of equilibrium state may be considered as a coincidence of $(k+1)$ different equilibrium states, of which the coincident state with the lowest number of boundaries [i.e. $(m-k)$] is the basic state.

4. STABILITY OF EQUILIBRIUM

The most general case of a one-way equilibrium state is that in which $m < n$ and all the terms $|\phi(V_r)|^j/|\phi|$ are positive. This is the case with which this section is primarily concerned.

The case in which $m = n$ and all terms $|\phi(V_r)|^j/|\phi|$ are positive is obviously stable and would require finite force to cause any small displacement from its equilibrium position.

The case of coincident equilibrium states is complicated by the fact that while the system is not held to its additional boundaries these must still be considered as rigid barriers to possible trial displacements. Obviously if we can show the basic equilibrium state to be stable without reference to any additional boundaries the imposition of these boundaries cannot make the equilibrium unstable. However, if the basic state is unstable the directions of perturbation which show instability may well lie entirely within the regions prohibited by the additional boundaries, in which case the equilibrium is still stable. In this paper the case of coincident equilibrium states will be dealt with briefly since its rigorous treatment would be algebraically extremely tiresome. The general principles of the treatment will however be set out later and it will be seen that it is quite feasible for suitably simple particular systems.

First, however, we consider structural systems having an equilibrium state for which $m < n$ and all $|\phi(V_r)|^j/|\phi|$ are positive.

We wish to determine whether an equilibrium state is stable, unstable, or critical. Returning to the power series (2) for the variation of potential energy about a fixed configuration, we have already specified that the first variation is zero for any displacements upon all m effective boundaries. Hence the dominant variation of potential energy for small displacements will be the second. Since such displacements are restricted to all m boundaries we use once again the amended form of potential energy,

$$W(Q_i, \Lambda) = V(Q_i, \Lambda) + \sum_{k=1}^{m} \lambda^k F^k(Q_i). \tag{19}$$

The first variation of $W(Q_i, \Lambda)$ is zero when λ^k has the value $-|\phi(V_r)|^k/|\phi|$, and we can write the second variation,

$$\delta^2 W = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2!}\left[V_{ij} - \sum_{k=1}^{m} \frac{|\phi(V_r)|^j}{|\phi|} F^k_{ij}\right] q_i q_j. \tag{20}$$

Before we can minimize this second variation for displacements restricted to the m boundaries it is necessary once again to amend the potential energy by introducing new

Lagrange multipliers μ^k. We call this further amended form of the potential energy $X(Q_i, \Lambda)$ where,

$$\delta^2 X = \sum_{i=1}^{n} q_i \left\{ \frac{1}{2!}\left[\sum_{j=1}^{n} V_{ij} q_j + \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda^k F_{ij}^k q_j \right] + \sum_{k=1}^{m} \mu^k F_i^k \right\}. \tag{21}$$

We can write this as,

$$\delta^2 X = \sum_{i=1}^{n} q_i \left\{ \frac{1}{2!} \sum_{j=1}^{n} W_{ij} q_j + \sum_{k=1}^{m} \mu^k F_i^k \right\}. \tag{22}$$

For stability we must show that the minimum value of $\delta^2 X$ is positive. First we obtain the derivatives of $\delta^2 X$ with respect to each of the n generalized coordinates, to locate the directions of extremum $\delta^2 X$,

$$\frac{\partial(\delta^2 X)}{\partial q_i} = \sum_{j=1}^{n} W_{ij} q_j + \sum_{k=1}^{m} \mu^k F_i^k = 0. \tag{23}$$

We now solve the first m of the equations for the Lagrange multipliers μ^k, which are,

$$\mu^k = \sum_{j=1}^{n} \left[-q_j \frac{|\phi(W_{rj})|^k}{|\phi|} \right]. \tag{24}$$

Substituting these values into (22) we obtain an expression for the second variation of the amended potential energy in all directions for which it is an extremum,

$$\delta^2 X_E = \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \left[W_{ij} - \sum_{k=1}^{m} W_{kj} \frac{|\phi(F_i^l)|_k}{|\phi|} \right]. \tag{25}$$

For stability of equilibrium we must show that $\delta^2 X_E$ is positive definite, subject still to the m constraints,

$$\sum_{i=1}^{n} F_i^j q_i = 0, \qquad j = 1, \dots, m. \tag{26}$$

If we solve (26) for the first m of the q_i in terms of the last $(n-m)$ we have,

$$q_r = -\sum_{t=(m+1)}^{n} q_t \frac{|\phi(F_t^j)|_r}{|\phi|}, \qquad r = 1, \dots, m. \tag{27}$$

We substitute these first m of the q_i into (25) and obtain an expression for $\delta^2 X_E$ in terms of the $(n-m)$ independent variables q_t. Omitting a certain amount of algebraic manipulation, this process gives,

$$\delta^2 X_E = \sum_{p=(m+1)}^{n} \sum_{t=(m+1)}^{n} \frac{1}{2!} X_{pt} q_p q_t, \tag{28}$$

in which X_{pt} is given by

$$X_{pt} = \left[W_{pt} - \sum_{r=1}^{m} W_{pr} \frac{|\phi(F_t^j)|_r}{|\phi|} - \sum_{r=1}^{m} W_{tr} \frac{|\phi(F_p^j)|_r}{|\phi|} + \sum_{r=1}^{m} \sum_{s=1}^{m} W_{rs} \frac{|\phi(F_p^j)|_r |\phi(F_t^j)|_s}{(|\phi|)^2} \right]. \tag{29}$$

The expression for $\delta^2 X_E$ given by (28) is then a homogeneous quadratic function of $(n-m)$ variables q_p with constant coefficients X_{pt} at a given configuration and load. The condition for such a function to be positive definite is that the determinant of the matrix of the coefficients X_{pt} is positive, and that each of its successively smaller principal minor determinants is also positive. In other words a total of $(n-m)$ determinants of the coefficients X_{pt}, of successively smaller dimensions and ending with a single coefficient $X_{(m+1)(m+1)}$, must be positive for stability of equilibrium. If we consider the diagonalization of the matrix $[X_{pt}]$ it can easily be seen that the whole determinant $|X_{pt}|$ must become zero at a critical equilibrium state at which stability is lost, while its principal minors may be either greater than or equal to zero.

We can now return briefly to the case of coincident equilibrium states, simply to formalize the process for determining the stability of particular cases. If we find in the equilibrium analysis that k of the m effective boundaries are in fact redundant due to their corresponding terms $|\phi(V_r)|^j/|\phi|$ being zero we can rewrite (28) for the "basic" equilibrium state which ignores these additional boundaries,

$$\delta^2 X_E = \sum_{p=(m+1-k)}^{n} \sum_{t=(m+1-k)}^{n} \frac{1}{2!} X_{pt} q_p q_t. \tag{28a}$$

Then, the stability condition is in the form of $(k+1)$ simultaneous inequalities,

$$\sum_{p=(m+1-k)}^{n} \sum_{t=(m+1-k)}^{n} X_{pt} q_p q_t > 0,$$

for all

$$\delta F^j = \sum_{i=1}^{n} F_i^j q_i \geq 0, \qquad j = (m-k), \ldots, m. \tag{30}$$

This condition will obviously only be tractable in simple cases with either very few generalized coordinates or a high degree of symmetry.

5. SIMPLE MODELS OF ONE-WAY SYSTEMS

We now use three simple two-degrees-of-freedom linkages to demonstrate the application of the theory in different situations. The first model will be more fully worked than the other two, to avoid repetition of basic algebra.

(a) *System with two coordinates and two boundaries*

The linkage is shown in Fig. 2. The joints at A, B, C, D, are pin-joints allowing rotation in the same plane, and the rotation at B and C is resisted by linear torsional springs of stiffness k. The springs are completely stress-free when the system is unloaded and $Q_1 = \frac{1}{2}d$, $Q_2 = \frac{1}{4}d$. There are two rigid barriers to the motion of B and C, each at a distance d from the loading-axis (where $d \ll L$). When the compressive load P is increased from zero both Q_1 and Q_2 will increase stably until Q_1 reaches its barrier. With further increase of load Q_2 continues to increase until it reaches its own barrier, after which no further displacement is possible and the load can increase infinitely. The role of the general theory is to confirm this intuitive observation in quantitative terms and to reveal any additional

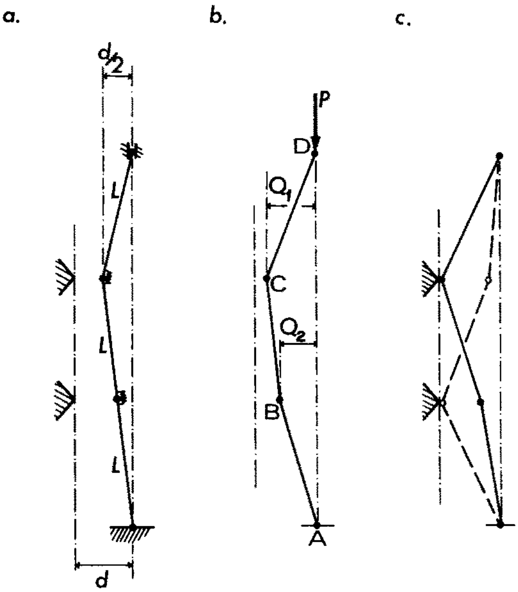

FIG. 2. Link-model with two boundaries. (a) Unloaded configuration. (b) With increasing load. (c) Other possible configurations.

equilibrium paths. The system has just two boundary functions $F^1 = (d-Q_1)$ and $F^2 = (d-Q_2)$, so that there are four possible distinct types of configuration. These are, a free configuration with both F^1 and F^2 greater than zero, the configurations with one of the boundary functions zero, and the configuration with both boundary functions zero.

The total potential energy for this model is,

$$V = \frac{k}{2L^2}[5Q_1^2 - 8Q_1Q_2 + 5Q_2^2 + \tfrac{3}{2}Q_2 d - 3Q_1 d + \tfrac{9}{16}d^2 - \Lambda(2Q_1^2 - 2Q_1Q_2 + 2Q_2^2 - \tfrac{3}{8}d^2)] + \text{higher order terms.} \tag{31}$$

The load-parameter Λ is (PL/k). The relevant derivatives are,

$$\left.\begin{aligned} V_1 &= \frac{k}{L^2}[Q_1(5-2\Lambda) + Q_2(\Lambda-4) - \tfrac{3}{2}d], \\ V_2 &= \frac{k}{L^2}[Q_1(\Lambda-4) + Q_1(5-2\Lambda) - \tfrac{3}{2}d], \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} V_{11} &= V_{22} = \frac{k}{L^2}(5-2\Lambda), \\ V_{12} &= \frac{k}{L^2}(\Lambda-4), \end{aligned}\right\} \tag{33}$$

$$F_1^1 = F_2^2 = -1, \qquad F_2^1 = F_1^2 = 0. \tag{34}$$

(i) *The free configuration.* If both F^1 and F^2 are positive the equilibrium conditions are $V_1 = 0$ and $V_2 = 0$. Solving these together we obtain,

$$\left.\begin{aligned} Q_1 &= \tfrac{3}{4}d(2-\Lambda)(\Lambda-3)^{-1}(\Lambda-1)^{-1}, \\ Q_2 &= \tfrac{3}{4}d(\Lambda-3)^{-1}(\Lambda-1)^{-1}. \end{aligned}\right\} \tag{35}$$

When Q_1 reaches its limiting value of d, either $\Lambda = 0{\cdot}56$ and $Q_2 = 0{\cdot}69d$, or $\Lambda = 2{\cdot}69$ and $Q_2 = -1{\cdot}44d$. For stability we know that both the whole stability determinant $|V_{ij}|$ and its principal minor V_{11} must be positive. This is only fulfilled for $\Lambda < 1$.

Equations (35) indicate three equilibrium paths, one stable and two unstable, which are shown in Fig. 3. Only the initial loading-path between $\Lambda = 0$ and $\Lambda = 0{\cdot}56$ is stable. The second unstable complementary path meets the boundary $F^2 = 0$ at $\Lambda = 3{\cdot}32, Q_1 = -1{\cdot}32d$.

(ii) *Configurations with* $F^1 = 0$. If $F^1 = 0$ with F^2 positive the equilibrium conditions are $|\Omega_2| = 0$ and $|\phi(V_r)|^1/|\phi| \geq 0$, where $|\phi| = F^1_1, |\phi(V_r)|^1 = V_1$ and,

$$|\Omega_2| = F^1_1 V_2 - F^1_2 V_1. \tag{36}$$

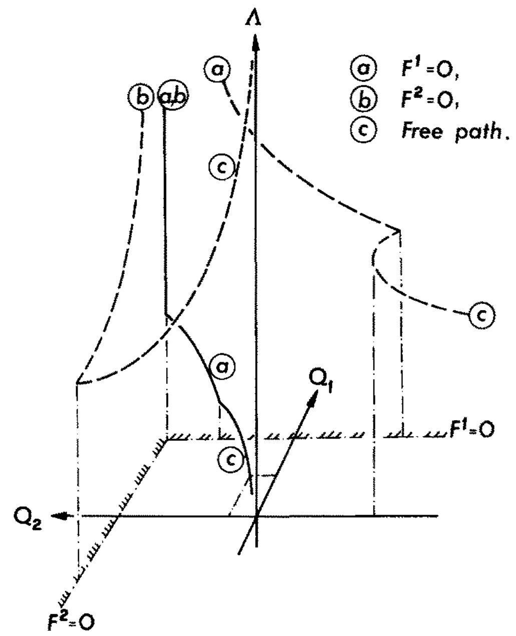

FIG. 3. Equilibrium paths of the model of Fig. 2. Unstable paths denoted by broken lines.

The former condition yields the solution,

$$Q_2 = d(3{\cdot}25-\Lambda)(5-2\Lambda)^{-1}, \tag{37}$$

which will only be valid when the second condition also holds. These regions are given by $Q_2 \geq 0{\cdot}69d$ and $-1{\cdot}44d \leq Q_2 \leq 0{\cdot}5d$. The two equilibrium paths are shown in Fig. 3, and join the free equilibrium paths where these meet the boundary $F_1 = 0$. The stability of the paths is indicated by the sign of the single element X_{22}, where,

$$X_{22} = V_{22} - 2V_{12}\frac{F_2^1}{F_1^1} + V_{11}\left(\frac{F_2^1}{F_1^1}\right)^2. \tag{38}$$

This indicates that the equilibrium is stable for $\Lambda < 2{\cdot}5$, confirming that the continuation of the initial free path on $F^1 = 0$ is itself stable, while its complementary path on $F^1 = 0$ which lies above $\Lambda = 2{\cdot}69$ is wholly unstable. The stable path meets $F^2 = 0$ at $\Lambda = 1{\cdot}75$.

(iii) *Configurations with* $F^2 = 0$. If $F^2 = 0$ and F^1 is positive the treatment is similar to (ii) above, giving just one, unstable, equilibrium path which meets an unstable free equilibrium path at $\Lambda = 3{\cdot}32$, $Q_1 = -1{\cdot}32d$.

(iv) *Configuration with* $F^1 = 0$, $F^2 = 0$. If $F^1 = 0$ and $F^2 = 0$ the equilibrium conditions are that $|\phi(V_r)|^1/|\phi| \geq 0$, and $|\phi(V_r)|^2/|\phi| \geq 0$, where,

$$\left.\begin{aligned} |\phi| &= F_1^1F_2^2 - F_2^1F_1^2, \\ |\phi(V_r)|^1 &= V_1F_2^2 - V_2F_1^2, \\ |\phi(V_r)|^2 &= F_1^1V_2 - F_2^1V_1. \end{aligned}\right\} \tag{39}$$

These conditions say simply that there is equilibrium if $\Lambda > 1{\cdot}75$, which must be stable.

(b) *System which exhibits loss of stability*

The linkage is shown in Fig. 4, and in order to demonstrate a loss of stability has initial symmetry of deformations. The initial value of both Q_1 and Q_2 is $\frac{1}{2}d$, and a single rigid point barrier is placed midway between B and C, at a distance d from the loading axis. We can deduce intuitively that as the load P is increased from zero both Q_1 and Q_2 increase stably at the same rate until the mid-point of BC touches the rigid barrier (we say that the barrier is so placed that it is exactly at the mid-point of BC in this deformed state rather than the initial state). There will be no change of Q_1 or Q_2 with further increase of load until a critical load is reached at which loss of stability occurs and the system buckles

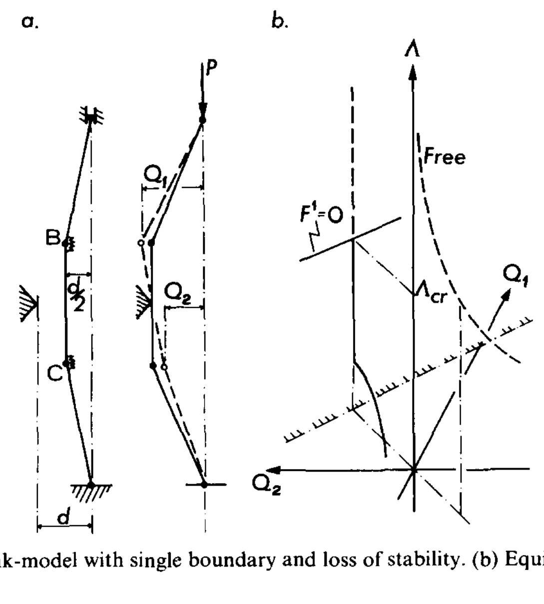

FIG. 4. (a) Link-model with single boundary and loss of stability. (b) Equilibrium paths.

asymmetrically about the centre-point of BC. The role of the general theory will now be to confirm our conclusions in quantitative terms, find the critical load, and discover any possible complementary paths.

The relevant relations are in this case,

$$\left.\begin{aligned} F^1 &= 2d - Q_1 - Q_2, \\ V &= \frac{k}{2L^2}[5Q_1^2 - 8Q_1Q_2 + 5Q_2^2 - Q_1d - Q_2d + \tfrac{1}{2}d^2 \\ &\quad - \Lambda(2Q_1^2 - 2Q_1Q_2 + 2Q_2^2 - \tfrac{1}{2}d^2)] + \ldots. \end{aligned}\right\} \tag{40}$$

The relevant derivatives of these are,

$$\left.\begin{aligned} V_1 &= \frac{k}{L^2}[Q_1(5-2\Lambda) + Q_2(\Lambda - 4) - \tfrac{1}{2}d], \\ V_2 &= \frac{k}{L^2}[Q_1(\Lambda - 4) + Q_2(5 - 2\Lambda) - \tfrac{1}{2}d], \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} V_{11} &= V_{22} = \frac{k}{L^2}(5-2\Lambda), \\ V_{12} &= \frac{k}{L^2}(\Lambda - 4), \end{aligned}\right\} \tag{42}$$

$$F_1^1 = F_2^1 = -1. \tag{43}$$

(i) *Free equilibrium*. Putting $V_1 = 0$ and $V_2 = 0$ for $F_1 \geq 0$ we find the free equilibrium equations,

$$Q_1 = Q_2 = \tfrac{1}{2}d(1-\Lambda)^{-1}. \tag{44}$$

Both Q_1 and Q_2 reach the value d simultaneously at $\Lambda = 0{\cdot}5$. Evaluating $|V_{ij}|$ and V_{11} we find that the initial path from $\Lambda = 0$ to $\Lambda = 0{\cdot}5$ is stable and its complementary path (Fig. 4) is unstable.

(ii) *Configurations with* $F^1 = 0$. The equilibrium conditions $|\Omega_2| = 0$ and $|\phi(V_r)|^1/|\phi|$ give,

$$(d - Q_1)(\Lambda - 3) = 0, \tag{45}$$

$$d(17 - 4\Lambda) + Q_1(6\Lambda - 18) \geq 0. \tag{46}$$

The equation (45) gives a "trivial" equilibrium path with $Q_1 = Q_2 = d$ for all Λ, or adjacent equilibrium states at $\Lambda = 3$. Putting $Q_1 = d$ in (46) gives us that the equilibrium is valid for $\Lambda \geq 0{\cdot}5$ Stability is again determined by the sign of X_{22}, which is,

$$X_{22} = \frac{6k}{L^2}(3 - \Lambda), \tag{47}$$

so that the equilibrium state loses stability at $\Lambda = 3$.

(c) *System with coincident equilibrium states*

The linkage is shown in Fig. 5. The connections at A, B, C, are now not pin-joints but universal-joints. The torsional spring of constant k_1 at A resists deflection of B in the direction of the Q_1-axis and the spring of constant k_2 at C resists deflection of B in the direction of the Q_2-axis, perpendicular to Q_1. In the unloaded, unstressed state ABC is straight. The potential energy of the system is given by,

$$V = Q_1^2\left(\frac{k_1}{2L^2}-\frac{P}{L}\right)+Q_2^2\left(\frac{k_2}{2L^2}-\frac{P}{L}\right)+\dots \tag{48}$$

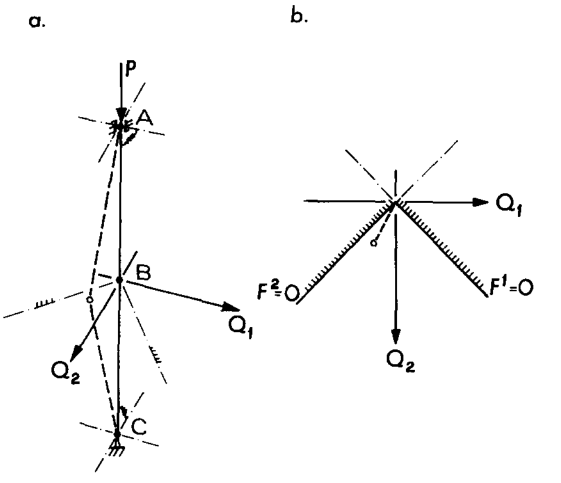

Fig. 5. (a) Link-model with coincident equilibrium states. (b) Plan view.

Two smooth rigid boundary surfaces are fixed in space at B, placed symmetrically about the Q_2-axis. The system's boundary functions due to these boundaries are,

$$\left.\begin{aligned} F_1 &= Q_1+Q_1, \\ F^2 &= Q_2-Q_1. \end{aligned}\right\} \tag{49}$$

It is evident that we have coincidence of the equilibrium states on either two boundaries or one boundary with a free equilibrium state. The problem is to show that this state is stable with increasing load until it reaches a discrete critical equilibrium state, and to determine the critical load.

The so-called "basic" equilibrium state is the free equilibrium state, so that for stability we must show that $\delta^2 V$ is positive in all incremental regions around the equilibrium point for which both δF^1 and δF^2 are positive or zero. In this case, $\delta^2 V = q_1^2(k_1/2L^2 - P/L) + q_2^2(k_2/2L^2 - P/L)$, so if we say k_1 is smaller than k_2 we know that $\delta^2 V$ is positive definite for P values below $(k_1/2L)$ and that $\delta^2 V$ is negative definite for P above $(k_2/2L)$. We have therefore to look in the range $k_1/2L < P < k_2/2L$ for our critical equilibrium state at which the system loses stability.

Consider a deflection from the equilibrium position into the permissible-space with respect to both boundaries. For such a displacement we know that $q_2 \geq q_1$ and $q_2 \geq -q_1$, so that,

$$\delta^2 V \geq q_1^2\left(\frac{k_1+k_2}{2L^2}-\frac{2P}{L}\right). \tag{50}$$

The inequality (50) obviously becomes the equation $\delta^2 V = q_1^2[(k_1+k_2)/2L^2-2P/L]$ for the limiting displacements along either of the boundaries. Hence we know that $\delta^2 V$ is positive definite within the prescribed region if $P < (k_1+k_2)/2L$, and that $\delta^2 V$ becomes zero along both boundaries if $P = (k_1+k_2)/2L$. This is then the critical load at which the equilibrium loses its stability. The direction of initial postbuckled motion will lie along either of the two boundaries.

6. CONCLUDING REMARKS

The main results from the above study may be summarized as follows:

1. For the equilibrium of a system of n degrees of freedom on m effective boundaries ($m < n$) we have,

$$|\Omega_p| = 0, \qquad i = (m+1), \ldots, n,$$

and

$$|\phi(V_r)|^j/|\phi| \geq 0, \qquad j = 1, \ldots, m.$$

Putting $m = 0$ these reduce to the usual conditions for a free system,

$$V_i = 0, \qquad i = 1, \ldots, n.$$

2. For stability of equilibrium we have,

$$|X_{pt}| > 0, \quad \text{and all principal minors} > 0, \qquad p, t = (m+1), \ldots, n.$$

Putting $m = 0$ these reduce to the usual conditions for a (non-diagonalized) free system,

$$|V_{ij}| > 0, \quad \text{and all principal minors} > 0, \qquad i, j = 1, \ldots, n.$$

It is hoped that these conditions will provide a starting point for future studies in the field of one-way systems. It has already been shown [11] that problems of an elastic ring loaded within a rigid circular cavity, and of the one-way case of the buckling of a beam on an elastic foundation, can be solved satisfactorily by discrete methods using the general theory given here.

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7. APPENDIX

Notation

For convenience in the handling of arrays containing large numbers of elements, often in determinant form, we use a specialized notation. This is mainly concerned with writing concisely the process for solving linear simultaneous equations.

(i) *General principles of notation.* Consider a matrix of dimensions $m \times m$, whose typical element placed in the ith column and jth row is denoted by A_{ij} (alternative forms may be A_j^i, A_i^j, A^{ij}). We shall denote this matrix as $[A_{ij}]$ and its determinant as $|A_{ij}|$. A vector of m elements b_j is described as $[b_j]$.

If the kth column of $[A_{ij}]$ is replaced by the vector $[b_j]$ we denote the resulting matrix as $[A_{ij}(b_j)]_k$ and its determinant as $|A_{ij}(b_j)|_k$. Similarly, if the lth row of $[A_{ij}]$ is replaced by the vector $[b_i]$, we denote the resulting matrix as $[A_{ij}(b_i)]^l$ and its determinant as $|A_{ij}(b_i)|^l$.

The significance of the above may be demonstrated by considering a system of m linear simultaneous equations in m independent variables. This system may be described as,

$$[A_{ij}] \, . \, [x_i] = [b_j].$$

The m solutions of these equations are given by,

$$x_k = \frac{|A_{ij}(b_j)|_k}{|A_{ij}|}.$$

(ii) *Specific notation used.* The most common array used in the general theory is $[F_s^r]$, the $m \times m$ matrix of the first m derivatives of the m effective boundary functions F_s^r, arranged so that the subscript s denotes the column, and the superscript r the row, of a given element. We call this matrix $[\phi]$ for convenience. The other arbitrarily defined array is $[\Omega_t]$ which is of dimensions $(m+1) \times (m+1)$. This is formed by adding to the square array $[\phi]$ an $(m+1)$th column $[F_t^r]$, where t is the integer subscript in $[\Omega_t]$, and an $(m+1)$th row $[V_s]$ where s runs from 1 to m. The element V_t is placed in the "corner" location $[(m+1), (m+1)]$. The subscript t can take any specific integer value between $(m+1)$ and n. The determinants of these two defined matrices are related by the property,

$$|\Omega_t| = V_t|\phi| - \sum_{i=1}^{m} V_i|\phi(F_t^j)|_i.$$

It should be noted that the elements of the substituted column vector $[F_t^j]$ each have an identical subscript t, being the subscript in $[\Omega_t]$, while j runs from 1 to m.

A further array $[X_{pt}]$ is used in the stability theory, and this has dimensions $(n-m)\times(n-m)$. The integer subscripts p and t each run from $(m+1)$ to n.

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Абстракт—Обсуждаются одномерные структурные системы в смысле обобщенных координат для дискретных систем. С этой точки зрения начинается общая методика обработки таких систем и установливаются усльоця для равновесия и устойчивости равновесия. Дается способ применения общей теории к частным проблемам, на основе трех простых структурных моделей с двумя степенями свободы.